

Invariant states of quantum birth and death chains

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A sufficient condition is given for a class of quantum birth and death chains on the non-negative integers to possess invariant states. The result is applied to generalised one-atom masers and to the Jaynes-Cummings one-atom maser with random interaction time and not necessarily diagonal atomic states.

1 Introduction

A classical birth and death chain on the non-negative integers is a homogeneous discrete Markov process $(x_t)_{t \in \mathbb{Z}}$ on the state space $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, where only nearest-neighbour-transitions occur. Such a process is determined by its birth rates $\lambda_n = P(x_t = n + 1 | x_{t-1} = n)$, for $n \in \mathbb{N}_0$, and death rates $\mu_n = P(x_t = n - 1 | x_{t-1} = n)$, for $n \in \mathbb{N}$. Suppose that $\mu_n \neq 0$ for all $n \in \mathbb{N}$, and put $\pi_n := \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}$, $\pi_0 := 1$. Then (x_t) has an invariant state if and only if the sequence $(\pi_n)_n$ is summable. In that case, the invariant state is unique and its density is given by $\rho_n = \frac{\pi_n}{\sum_n \pi_n}$. In particular, (x_t) has an invariant state whenever there exists a constant c with $\frac{\lambda_n}{\mu_{n+1}} \leq c < 1$ for almost all $n \in \mathbb{N}$.

In the present paper, we consider invariant states of quantum birth and death chains (QBDCs). For us, a QBDC is a quantum Markov chain on $\mathcal{B}(\ell^2(\mathbb{N}_0))$ in the sense of a unital completely positive (ucp) map $T : \mathcal{B}(\ell^2(\mathbb{N}_0)) \rightarrow \mathcal{B}(\ell^2(\mathbb{N}_0))$, with the extra-condition that only nearest-neighbour transitions are allowed. This means that the transition rates $\text{Tr}(e_{n,m} T(e_{k,l}))$, where $e_{n,m}, e_{k,l} \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ denote matrix units, vanish whenever $|m - k| > 1$ or $|n - l| > 1$. Examples of QBDCs are provided by the one-atom maser as studied in, e.g. [BP09, Bru13], and by its generalisations considered in [BGK⁺13]. QBDCs are fundamentally different from “unitary” or “open quantum random walks” as studied in, e.g. [Kon02] or [APSS12]. States for such quantum random walks not only specify a “position” on \mathbb{N} (or \mathbb{Z}), but in addition specify the state of a “coin”. Open quantum random walks where the space of states of the coin is trivial (i.e. \mathbb{C}) are in fact classical Markov chains.

In addition to the analogues of the classical birth and death rates $\lambda_n = \text{Tr}(e_{n,n}T(e_{n+1,n+1}))$ and $\mu_n = \text{Tr}(e_{n,n}T(e_{n-1,n-1}))$, a QBDC is characterised, among others, by the transition rates $\eta_n = \text{Tr}(e_{n+1,n}T(e_{n,n}))$, see Figure 1. In this essentially two-dimensional setting it seems no more possible to give an explicit formula for an invariant state, as it was in the classical case. Moreover, the condition $\frac{\lambda_n}{\mu_{n+1}} \leq c < 1$ is no more sufficient to guarantee the existence of an invariant state (see Section 4 for an example). However, a simple sufficient criterion can be given which involves the parameters λ_n, μ_n and η_n only. Namely, if $\kappa := \liminf_n \frac{1}{n} \ln(\pi_n^{-1}) > 0$ and

$$\liminf_n \frac{\lambda_n \mu_{n+1}}{4|\eta_n|^2} > \frac{e^{-\kappa}}{(1 - e^{-\kappa})^2} , \quad (1)$$

then T has an invariant state (Theorem 7). The proof combines ideas from [FR01] with a positivity criterion for tri-diagonal matrices.

While previous works have often focused on QBDCs with $\eta_n = 0$ for $n \gg 0$, our result is suitable for QBDCs with $\eta_n \neq 0$. We apply it to the class of QBDCs, inspired by the micromaser experiment, which were considered in [BGK⁺13], and to the Jaynes-Cummings one-atom maser with random interaction time. In the latter case it is shown, in particular, that $\frac{\lambda_n}{\mu_{n+1}} \leq c < 1$ suffices to guarantee the existence of invariant states even for non-thermal (non-diagonal) and non-pure atomic states. Moreover, for non-pure atomic states, every initial state approaches the invariant state (Proposition 11).

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2 Notational preliminaries

The set of natural numbers is taken as $\mathbb{N} := \{1, 2, 3, \dots\}$. Scalar products $\langle \cdot, \cdot \rangle$ are linear in the first slot and anti-linear in the second. If \mathcal{H} is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on \mathcal{H} . A unital completely positive linear map $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ will be referred to as a *ucp-map*. The canonical basis of $\ell^2(\mathbb{N}_0)$ is denoted by $(e_n)_{n \in \mathbb{N}_0}$. For $n, m \in \mathbb{N}_0$, the *matrix unit* $e_k \mapsto \delta_{m,k} e_n$ is denoted by $e_{n,m}$, and the orthogonal projection onto $\text{span}\{e_k \mid n \leq k \leq m\} \subset \ell^2(\mathbb{N}_0)$ by $p_{[n,m]}$.

Definition 1. A *quantum birth and death chain* (on the non-negative integers; QBDC) is a ucp-map $T : \mathcal{B}(\ell^2(\mathbb{N}_0)) \rightarrow \mathcal{B}(\ell^2(\mathbb{N}_0))$ satisfying

$$p_{[n+1,m-1]} \leq T(p_{[n,m]}) \leq p_{[n-1,m+1]} \quad (2)$$

for all $n, m \in \mathbb{N}_0$ with $n \leq m$. Here, $p_{[-1,m+1]}$ is understood to be $p_{[0,m+1]}$.

For a quantum birth and death chain T we introduce the *transition rates*

$$\begin{aligned} \sigma_n &:= \text{Tr}(e_{n,n}T(e_{n,n})) , & \mu_n &:= \text{Tr}(e_{n,n}T(e_{n-1,n-1})) , \\ \lambda_n &:= \text{Tr}(e_{n,n}T(e_{n+1,n+1})) , & \eta_n &:= \text{Tr}(e_{n+1,n}T(e_{n,n})) . \end{aligned} \quad (3)$$

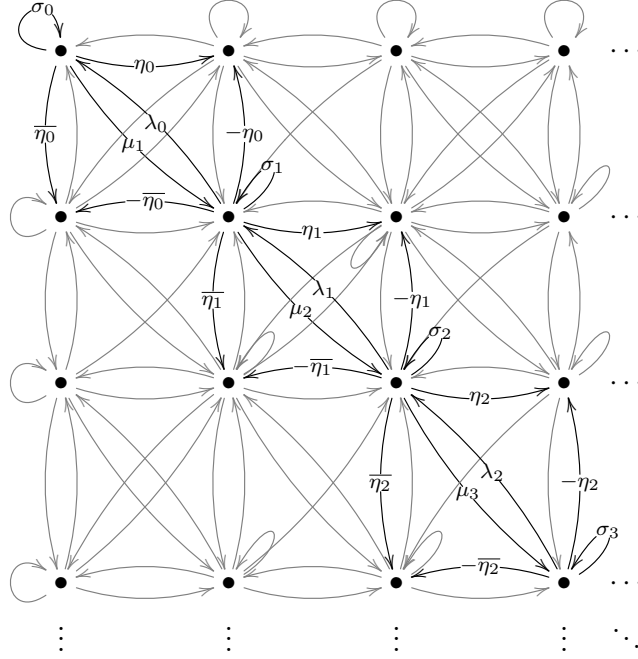


Figure 1: Transition rates for a QBDC. An arrow from the point (m, m') to (n, n') on the lattice $\mathbb{N}_0 \times \mathbb{N}_0$ indicates that the transition rate $\text{Tr}(e_{n',n}T(e_{m,m'}))$ for a QBDC T not necessarily vanishes.

The transition rates $\text{Tr}(e_{n',n}T(e_{m,m'}))$ vanish due to (2) whenever $|n - m| > 1$ or $|n' - m'| > 1$. Unitality of T implies that $\text{Tr}(e_{n,n}T(e_{n-1,n+1})) = 0 = \text{Tr}(e_{n,n}T(e_{n+1,n-1}))$, as well as the relations $1 = \sigma_n + \lambda_n + \mu_n$ and $\eta_n = -\text{Tr}(e_{n+1,n}T(e_{n+1,n+1}))$; positivity gives $\overline{\eta}_n = \text{Tr}(e_{n,n+1}T(e_{n,n})) = -\text{Tr}(e_{n+1,n}T(e_{n+1,n+1}))$. If we draw an arrow from (m, m') to (n, n') on the lattice $\mathbb{N}_0 \times \mathbb{N}_0$ and label it with the transition rate $\text{Tr}(e_{n',n}T(e_{m,m'}))$ whenever it does not vanish, then a generic QBDC T may be visualised by a diagram as in Figure 1.

For an unbounded operator X with domain $\text{Dom}(X) = D := \text{span}\{e_k \mid k \in \mathbb{N}_0\}$ let $X_{n,m} := \langle Xe_m, e_n \rangle$. If T is a QBDC, then

$$T(X) := \text{strong-lim}_{N \in \mathbb{N}_0} \sum_{0 \leq n, m \leq N} X_{n,m} \cdot T(e_{n,m}) \quad (4)$$

defines another operator with domain D .

3 Existence of normal invariant states

We investigate the existence of normal invariant states using similar techniques as applied in [FR01] to the case of a continuous-time-semigroup with a form-generator. The case of a semigroup in discrete time as considered here, i.e. of powers of a single ucp-map, is much less technically involved, of course.

Lemma 2 (cf. [FR01, Prop. II.1]). *Let φ be a normal state on a Hilbert space \mathcal{H} and T a ucp-map on $\mathcal{B}(\mathcal{H})$. Then all weak cluster points of the sequence*

$$\frac{1}{n} \sum_{k=1}^n \varphi \circ T^k \quad (5)$$

in the predual of $\mathcal{B}(\mathcal{H})$ are normal invariant states for T .

Definition 3. A sequence $(\varphi_n)_{n \in \mathbb{N}_0}$ of normal states on $\mathcal{B}(\mathcal{H})$ is called *tight*, if for each $\varepsilon > 0$ there exist a finite rank projection $p \in \mathcal{B}(\mathcal{H})$ and $N \in \mathbb{N}_0$, such that

$$\varphi_n(p) > 1 - \varepsilon \quad (6)$$

for all $n > N$.

Lemma 4 ([FR01, Thm. II.1]; see also [Haa06, Lem. 2.2.4]). *Let $(\varphi_n)_{n \in \mathbb{N}_0}$ be a tight sequence of normal states on $\mathcal{B}(\mathcal{H})$. Then $(\varphi_n)_{n \in \mathbb{N}_0}$ possesses a weak cluster point in the predual of $\mathcal{B}(\mathcal{H})$.*

Given a self-adjoint operator Y and a measurable subset $A \subseteq \mathbb{R}$, denote Y 's spectral projection corresponding to A by $\chi_A(Y)$. For a one-parameter-semigroup version of the following lemma – without the fall-off statement, see [FR01, Thm. II.1].

Lemma 5. *Let T be a QBDC. Let X, Y be self-adjoint operators on $\ell^2(\mathbb{N}_0)$ with $D \subseteq \text{Dom}(X), \text{Dom}(Y)$, and assume that X is positive, Y is bounded from below by $-b$ ($b > 0$) and Y 's spectral projections associated to bounded sets are finite dimensional. If*

$$\sum_{k=1}^n \langle T^k(Y)\xi, \xi \rangle \leq \langle X\xi, \xi \rangle \quad (7)$$

for all $n \in \mathbb{N}$ and $\xi \in D$, then T has a normal invariant state φ such that $\varphi(\chi_{[c,d]}(Y)) \leq \frac{b}{c}$ holds for all $0 < c \leq d$.

Proof. Observe that for any (unbounded) operator A with domain $D \subseteq \text{Dom}(A)$ which satisfies $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in D$, one has $\langle T(A)\xi, \xi \rangle \geq 0$ for all $\xi \in D$: Namely, with $M \geq 0$ large enough such that $\xi = p_{[0,M]}\xi$,

$$\langle T(A)\xi, \xi \rangle = \langle p_{[0,M]}T(A)p_{[0,M]}\xi, \xi \rangle \stackrel{(2) \& (4)}{=} \langle p_{[0,M]}T(p_{[0,M+1]}Ap_{[0,M+1]})p_{[0,M]}\xi, \xi \rangle \geq 0. \quad (8)$$

Iterating this, one obtains $\langle T^n(A)\xi, \xi \rangle \geq 0$ for all $\xi \in D$ and all $n \in \mathbb{N}$.

Now we have for each $r > 0$

$$Y \geq -b\chi_{(-\infty, r]}(Y) + r\chi_{(r, \infty)}(Y) = -(b+r)\chi_{(-\infty, r]}(Y) + r\mathbb{1}. \quad (9)$$

So,

$$-(b+r) \sum_{k=1}^n \langle T^k(\chi_{(-\infty, r]}(Y))\xi, \xi \rangle + nr\|\xi\|^2 \leq \langle X\xi, \xi \rangle. \quad (10)$$

Bringing $nr\|\xi\|^2$ to the other side and dividing by $-(b+r)n$ gives

$$\frac{1}{n} \sum_{k=1}^n \langle T^k(\chi_{(-\infty, r]}(Y))\xi, \xi \rangle \geq \frac{r\|\xi\|^2}{b+r} - \frac{\langle X\xi, \xi \rangle}{n(b+r)}. \quad (11)$$

Choosing ξ with $\|\xi\| = 1$, the last line says that the sequence of states $\frac{1}{n} \sum_{k=1}^n T_*^k(|\xi\rangle\langle\xi|)$ is tight, since $\chi_{(-\infty, r]}(Y) = \chi_{[-b, r]}(Y)$ is a finite rank projection (here, T_* denotes the predual of the map $T : \mathcal{B}(\ell^2(\mathbb{N}_0)) \rightarrow \mathcal{B}(\ell^2(\mathbb{N}_0))$).

Let φ be a weak cluster point of that sequence according to Lemma 4. By Lemma 2, φ is a normal invariant state. From $Y \geq -b\mathbb{1} + c\chi_{[c, d]}(Y)$ and (7) we get

$$-nb + c \sum_{k=1}^n \langle T^k(\chi_{[c, d]}(Y))\xi, \xi \rangle \leq \langle X\xi, \xi \rangle, \quad (12)$$

which shows the estimate $\varphi(\chi_{[c, d]}(Y)) \leq \frac{b}{c}$. \square

See [FR01, Thm. IV.1] for a one-parameter-semigroup version of the following proposition.

Proposition 6. *Let T be a QBDC. Let X, Y be self-adjoint (unbounded) operators on $\ell^2(\mathbb{N}_0)$ such that $D \subseteq \text{Dom}(X), \text{Dom}(Y)$, X is positive and Y is bounded from below. Assume that Y 's spectral projections associated to bounded sets are finite dimensional. If*

$$\langle (T(X) - X)\xi, \xi \rangle \leq -\langle Y\xi, \xi \rangle \quad (13)$$

for all $\xi \in D$, then T has a normal invariant state φ such that $\varphi(\chi_{[c, d]}(Y)) \leq \frac{b}{c}$ holds for all $0 < c \leq d$ and some $b > 0$.

Proof. We have, for all $\xi \in D$,

$$\langle X\xi, \xi \rangle \geq \langle (X - T^{n+1}(X))\xi, \xi \rangle = \sum_{k=0}^n \langle T^k(X - T(X))\xi, \xi \rangle \geq \sum_{k=0}^n \langle T^k(Y)\xi, \xi \rangle. \quad (14)$$

Now the claim follows on using Lemma 5. \square

We come to the main result. If $\mu_k \neq 0$ for all k , then put $\pi_n := \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$. A normal state φ on $\mathcal{B}(\ell^2(\mathbb{N}_0))$ is said to be of *exponential fall-off* if there are constants $C, \gamma > 0$ such that $\varphi(e_{n,n}) \leq Ce^{-\gamma n}$ for all $n \in \mathbb{N}_0$.

Theorem 7. *Let T be a QBDC and let μ_n, λ_n be as in (3).*

1. *Suppose that $\lambda_k, \mu_k \neq 0$ for all k and that $\kappa := \liminf_n \frac{1}{n} \ln(\pi_n^{-1}) > 0$. If*

$$\liminf_n \frac{\lambda_n \mu_{n+1}}{4|\eta_n|^2} > \frac{e^{-\kappa}}{(1 - e^{-\kappa})^2}, \quad (15)$$

then T has a normal invariant state of exponential fall-off.

2. Suppose that $\lambda_n \neq 0$ for all $n \in \mathbb{N}$ and that there is $m \in \mathbb{N}$ such that $\mu_n < \lambda_n$ for all $n > m$. If

$$\liminf_n \frac{(\lambda_n - \mu_n)(\lambda_{n+1} - \mu_{n+1})}{4|\eta_n|^2} > 1, \quad (16)$$

then T has no normal invariant state.

Proof. Part 1: Choose $e^{-\kappa} < t < 1$ and $0 < r < 1 - t$ such that still

$$\frac{\lambda_n \mu_{n+1}}{4|\eta_n|^2} > \frac{t}{(1 - r - t)^2} \quad (17)$$

holds for all $n > N$, with some $N \in \mathbb{N}$. Let $X := \text{diag}(x_0, x_1, x_2, \dots)$ and $Y := \text{diag}(y_0, y_1, y_2, \dots)$, where $x_n := \sum_{k=1}^n (\mu_k \pi_k)^{-1} t^k$ for $n \in \mathbb{N}_0$ and $y_n := r \pi_n^{-1} t^n$ for $n > N$ (the y_n 's for $n = 1, 2, \dots, N$ will be chosen later); the domains of X and Y are taken as $D(X) = D(Y) = D := \text{span}\{e_n : n \in \mathbb{N}_0\} \subset \ell^2(\mathbb{N}_0)$. Since $y_n = r \cdot (\pi_n^{-1} e^{-n\kappa}) \cdot (te^\kappa)^n \rightarrow \infty$, Y 's spectral projections associated to bounded sets are finite dimensional. Both X and Y are densely defined and semi-bounded symmetric operators, hence, they possess self-adjoint extensions. We have

$$(T(X))_{n,n} = \sigma_n x_n + \lambda_n x_{n+1} + \mu_n x_{n-1} \quad (18)$$

for $n \geq 1$, $(T(X))_{0,0} = \lambda_0 x_1 = \lambda_0 t$, and

$$\begin{aligned} (T(X))_{n,1+n} &= \eta_n(x_n - x_{n+1}), & (T(X))_{1+n,n} &= \overline{(T(X))_{n,1+n}}, \\ (T(X))_{n,k+n} &= (T_\psi(X))_{k+n,n} = 0, \end{aligned} \quad (19)$$

for all $n \geq 0$ and $k > 1$, i.e. $T(X)$ is a tridiagonal operator. Using $1 = \sigma_n + \lambda_n + \mu_n$, we obtain

$$\begin{aligned} (T(X) - X)_{n,n} &= \lambda_n(x_{n+1} - x_n) - \mu_n(x_n - x_{n-1}) = \frac{\lambda_n}{\mu_{n+1}\pi_{n+1}} t^{n+1} - \pi_n^{-1} t^n \\ &= \pi_n^{-1} (t - 1) t^n \end{aligned} \quad (20)$$

for all $n \geq 1$. We want to show that $T(X) - X \leq -Y$, i.e. $0 \leq X - T(X) - Y$, because then the existence of a normal invariant state follows from Proposition 6. To this end, by [BCSW78, Prop. 1] and since $X - Y - T(X)$ is tridiagonal and symmetric, it suffices to show that

- 1) the diagonal entries of $X - T(X) - Y$ are positive,
- 2) the following expressions are positive:

$$\begin{aligned} &(X - T(X) - Y)_{n,n} (X - T(X) - Y)_{n+1,n+1} \\ &- 4(X - T(X) - Y)_{n+1,n} (X - T(X) - Y)_{n,n+1}. \end{aligned} \quad (21)$$

Firstly,

$$(X - T(X) - Y)_{n,n} = \pi_n^{-1}(1-t)t^n - \pi_n^{-1}rt^n = \pi_n^{-1}(1-r-t)t^n > 0 \quad (22)$$

for all $n > N$. Secondly, for all $n > N$ we have

$$\begin{aligned} & (X - T(X) - Y)_{n,n}(X - T(X) - Y)_{n+1,n+1} \\ & - 4(X - T(X) - Y)_{n+1,n}(X - T(X) - Y)_{n,n+1} \\ & = \pi_n^{-1}(1-r-t)t^n \cdot \pi_{n+1}^{-1}(1-r-t)t^{n+1} - 4|\eta_n|^2 \frac{t^{2n+2}}{\mu_{n+1}^2 \pi_{n+1}^2} \\ & = \frac{1}{\mu_{n+1}^2 \pi_{n+1}^2} (\lambda_n \mu_{n+1} (1-r-t)^2 - 4|\eta_n|^2 t) t^{2n+1} \stackrel{(17)}{>} 0. \end{aligned} \quad (23)$$

Finally, choosing $y_n \in \mathbb{R}$ sufficiently small ($y_n \ll 0$) for $n = 1, 2, \dots, N$, both conditions 1) and 2) can be fulfilled for all $n \in \mathbb{N}_0$.

As Y is diagonal with exponentially growing eigenvalues y_n , the fall-off statement in Proposition 6 shows that T has a normal invariant state of exponential fall-off.

Part 2: First, from the assumption $\lambda_n \neq 0$ it follows that for any invariant normal state φ of T there is $m \in \mathbb{N}$ such that $\varphi(e_{n,n}) \neq 0$ for all $n \geq m$: For, if $\varphi(e_{n+1,n+1}) = 0$, then, by positivity of φ we also have $\varphi(e_{n,n+1}) = 0 = \varphi(e_{n+1,n})$. Hence, if $\varphi(e_{n,n}) \neq 0$ but $\varphi(e_{n+1,n+1}) = 0$, then $\varphi(T(e_{n+1,n+1})) = \lambda_n \varphi(e_{n,n}) + \mu_n \varphi(e_{n+2,n+2}) > 0$, in contradiction to φ being invariant.

Now it suffices to find a sequence z_0, z_1, z_2, \dots of real numbers, a sequence $\varepsilon_0, \varepsilon_1, \dots \geq 0$ and a constant $C > 0$ with the following properties:

- 1) $\varepsilon_n \neq 0$ for infinitely many $n \in \mathbb{N}$,
- 2) $|z_{n+1} - z_n| \leq C$ for all $n \in \mathbb{N}$,
- 3) $\langle (T(Z) - Z - \varepsilon)\xi, \xi \rangle \geq 0$ for all $\xi \in D$, where Z denotes the (unbounded) operator $Z := \text{diag}(z_0, z_1, z_2, \dots)$, and $\varepsilon := \text{diag}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$.

Namely, if φ is a normal invariant state on $\mathcal{B}(\ell^2(\mathbb{N}_0))$, choose $n, m \in \mathbb{N}$, $m > n$, with $\varphi(p_{[0,m]}^\perp) < (\frac{\delta}{18C})^2$ for $\delta := \varepsilon_n \varphi(e_{n,n}) > 0$. With $Z^{\wedge m} := \text{diag}(z_0, z_1, \dots, z_{m-1}, z_m, z_m, \dots) \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ we have $\|T(Z^{\wedge m}) - Z^{\wedge m}\| \leq 3C$, since $T(Z^{\wedge m}) - Z^{\wedge m}$ is tridiagonal with entries bounded by C (note that $0 \leq \lambda_k, \mu_k, |\eta_k| \leq 1$). Hence,

$$\begin{aligned} \varphi(T(Z^{\wedge m+1}) - Z^{\wedge m+1}) &= \underbrace{\varphi(p_{[0,m]}(T(Z^{\wedge m+1}) - Z^{\wedge m+1})p_{[0,m]})}_{\geq \varepsilon_n \varphi(e_{n,n}), \text{ by 3)}} \\ &+ \underbrace{\varphi(p_{[0,m]}^\perp(T(Z^{\wedge m+1}) - Z^{\wedge m+1})p_{[0,m]})}_{\leq \sqrt{\varphi(p_{[0,m]}^\perp)} \|T(Z^{\wedge m+1}) - Z^{\wedge m+1}\|} + \underbrace{\varphi(p_{[0,m]}(T(Z^{\wedge m+1}) - Z^{\wedge m+1})p_{[0,m]}^\perp)}_{\leq \sqrt{\varphi(p_{[0,m]}^\perp)} \|T(Z^{\wedge m+1}) - Z^{\wedge m+1}\|} \\ &+ \underbrace{\varphi(p_{[0,m]}^\perp(T(Z^{\wedge m+1}) - Z^{\wedge m+1})p_{[0,m]}^\perp)}_{\leq \sqrt{\varphi(p_{[0,m]}^\perp)} \|T(Z^{\wedge m+1}) - Z^{\wedge m+1}\|} > \delta - 3 \cdot \frac{\delta}{18C} \cdot 3C > 0, \end{aligned} \quad (24)$$

such that φ cannot be invariant.

Let us turn to the choice of Z . First, let $N \in \mathbb{N}_0$ be large enough such that for all $n > N$ there are $\varepsilon_n > 0$ with $\lambda_n - \mu_n - \varepsilon_n > 0$ and

$$\frac{(\lambda_n - \mu_n - \varepsilon_n)(\lambda_{n+1} - \mu_{n+1} - \varepsilon_{n+1})}{4|\eta_n|^2} > 1 . \quad (25)$$

Then define

$$Z := \text{diag}(\underbrace{0, 0, \dots, 0}_{N+2 \text{ times}}, 1, 2, 3, \dots) \quad \text{and} \quad \varepsilon := \text{diag}(\underbrace{0, 0, \dots, 0}_{N+1 \text{ times}}, \varepsilon_N, \varepsilon_{N+1}, \dots) . \quad (26)$$

As we have, for $n > N$,

$$\begin{aligned} (T(Z) - Z - \varepsilon)_{n,n} &= \lambda_n(z_{n+1} - z_n) - \mu_n(z_n - z_{n-1}) - \varepsilon_n = \lambda_n - \mu_n - \varepsilon_n , \\ (T(Z) - Z - \varepsilon)_{n,1+n} &= \eta_n(z_{n+1} - z_n) = \eta_n = \overline{(T(Z) - Z - \varepsilon)_{1+n,n}} , \end{aligned} \quad (27)$$

and $(T(Z) - Z - \varepsilon)_{n,m} = 0$ otherwise, we see that $T(Z) - Z \geq \varepsilon$. \square

4 Example: Generalised one-atom masers

Let ψ be a state on M_2 , parametrised by $0 \leq \lambda \leq 1$ and $\zeta \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ via

$$\psi(x) = \text{Tr} \left(\begin{pmatrix} \lambda & i\bar{\nu} \\ -i\nu & 1 - \lambda \end{pmatrix} x \right) , \quad x \in M_2 , \quad \text{where } \nu := i\sqrt{\lambda(1-\lambda)}\zeta . \quad (28)$$

Let $a = \text{diag}(1, \alpha_1, \alpha_2, \dots)$ and $b = \text{diag}(0, \beta_1, \beta_2, \dots)$ be infinite diagonal matrices with $-1 \leq \alpha_n, \beta_n \leq 1$ ($n \in \mathbb{N}$) and $a^2 + b^2 = \mathbb{1}$. So a, b give rise to bounded operators on $\ell^2(\mathbb{N}_0)$. With $s \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ denoting the right-shift $e_n \mapsto e_{n+1}$, let

$$\begin{aligned} T_\psi(x) &:= \lambda(s^* a s x s^* a s + s^* b x b s) + (1 - \lambda)(b s x s^* b + a x a) \\ &\quad - \bar{\nu}(a x b s - b s x s^* a s) + \nu(s^* a s x s^* b - s^* b x a) , \quad x \in \mathcal{B}(\ell^2(\mathbb{N}_0)) , \end{aligned} \quad (29)$$

be the transition operator on $\mathcal{B}(\ell^2(\mathbb{N}_0))$ associated with ψ , cf. [BGK⁺13]. Then T_ψ defines a QBDC, which can be regarded as a generalisation of the Jaynes-Cummings one-atom-maser. The transition rates of T_ψ are shown in Figure 2.

For this class of QBDCs one obtains, as a consequence of Theorem 7:

Proposition 8. *Let $\hat{\beta} := \limsup_n |\beta_n| > 0$, $\check{\beta} := \liminf_n |\beta_n| > 0$ and put $\hat{\alpha} := \sqrt{1 - \hat{\beta}^2}$, $\check{\alpha} := \sqrt{1 - \check{\beta}^2}$. We have:*

1. *If $\lambda < \frac{1}{2} - \frac{\hat{\alpha}}{\check{\beta}}|\nu|$, then T_ψ has a normal invariant state of exponential fall-off.*
2. *If $\lambda > \frac{1}{2} + \frac{\check{\alpha}}{\check{\beta}}|\nu|$, then T_ψ admits no normal invariant state.*

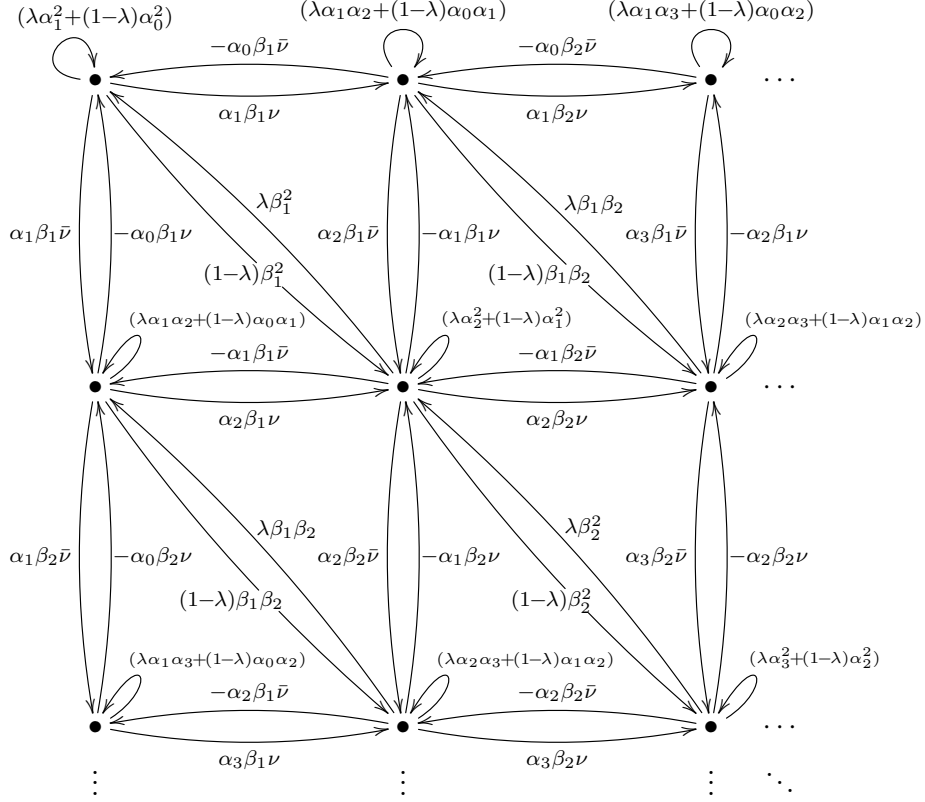


Figure 2: Action of T_ψ , where $i\zeta\sqrt{\lambda(1-\lambda)}$ is abbreviated by ν ; cf. [BGK⁺13, Fig. III.1].

Proof. For T_ψ , the coefficients λ_n, μ_n, η_n are given by $\lambda_n = \lambda\beta_{n+1}^2$, $\mu_n = (1-\lambda)\beta_n^2$, $\eta_n = -\alpha_n\beta_{n+1}\bar{\nu}$, as one may read off from Figure 2 or (29). Therefore, $\kappa = \liminf \ln \frac{\mu_n}{\lambda_{n-1}} = \ln \frac{1-\lambda}{\lambda}$, which is bigger than 0 if and only if $\lambda < \frac{1}{2}$. Now the right-hand side of (15) reads $\frac{e^{-\kappa}}{(1-e^{-\kappa})^2} = \frac{\frac{\lambda}{1-\lambda}}{(1-\frac{\lambda}{1-\lambda})^2} = \frac{\lambda(1-\lambda)}{(1-2\lambda)^2}$. For the left-hand side we obtain $\frac{\lambda_n\mu_{n+1}}{4|\eta_n|^2} = \frac{\lambda\beta_{n+1}^2(1-\lambda)\beta_n^2}{4\alpha_n^2\beta_{n+1}^2|\nu|^2} = \frac{\lambda(1-\lambda)\beta_n^2}{4|\nu|^2\alpha_n^2}$, which can be estimated from below by $\frac{\lambda(1-\lambda)\beta^2}{4|\nu|^2\alpha^2}$, for n sufficiently large. Now the first statement easily follows. The second statement is obtained from part 2 of Theorem 7 by a similarly easy calculation. \square

It is not known to the author how sharp condition (15) is. To address this question we consider the special case where $\alpha_n = \alpha$, $\beta_n = \beta$ are constant. This is referred to as “toy model” in [BGK⁺13]. Remarkably, for this toy model, for pure atomic states ψ and for $\alpha > 0$, the condition $\lambda < \frac{1}{2} - |\frac{\alpha}{\beta}\nu|$ in Proposition 8.1 marks the full set of parameters (λ, ζ) with $\lambda < \frac{1}{2}$ for which a *pure* invariant state exists: according to [BGK⁺13, Prop. 5.1], a pure invariant state exists in this case if and only if $\lambda < \frac{1}{2}(1-\alpha)$ holds. With $|\zeta| = 1$

(which amounts for the atomic state ψ being pure), one finds

$$\begin{aligned} \lambda < \frac{1}{2} - \left| \frac{\alpha}{\beta} \nu \right| = \frac{1}{2} - \left| \frac{\alpha}{\beta} \right| \lambda (1 - \lambda) &\iff \lambda^2 - \lambda + \underbrace{\frac{|\beta|^2}{4}}_{=\frac{1-|\alpha|^2}{4}} > 0 \\ \iff \left(\lambda - \frac{1-|\alpha|}{2} \right) \left(\lambda - \frac{1+|\alpha|}{2} \right) > 0. \end{aligned} \quad (30)$$

For $0 < \lambda < \frac{1}{2}$ and $-1 \leq \alpha \leq 1$, the second factor on the left-hand side in (30) is negative, hence, (30) is equivalent to $\lambda < \frac{1}{2}(1 - |\alpha|)$. If $-1 < \alpha < 0$, then there exist pure states ψ in the upper Bloch hemisphere ($\lambda > \frac{1}{2}$) for which T_ψ admits an invariant state. These are obviously not captured by Theorem 7.

The following proposition extends the non-existence statement in Proposition 8 and allows to strengthen the above observations. If $\alpha > 0$, it determines a parameter region in the lower Bloch hemisphere where the toy model does not admit invariant states.

Proposition 9. *Let $\alpha_n = \alpha$ and $\beta_n = \beta$ ($n > 0$) with $-1 < \alpha, \beta < 1$ and $\alpha^2 + \beta^2 = 1$. For $0 < \lambda < 1$, if $\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}}$ and $\frac{1-\alpha}{|\beta|} < \frac{|\nu|}{1-\lambda}$, then T_ψ has no normal invariant state.*

Proof. The idea is to construct, for any given normal state, an observable whose expectation value under this state strictly increases or decreases. These observables will be built from the “number operator” $N = \text{diag}(0, 1, 2, 3, \dots)$, which is an unbounded operator on $\ell^2(\mathbb{N}_0)$ with domain $D = \text{span}\{e_n \mid n \in \mathbb{N}_0\}$, or rather from its bounded truncations $N^{\wedge m} := \text{diag}(0, 1, \dots, m-1, m, m, m, \dots)$ for $m \geq 0$ and $N^{\wedge m} := 0$ for $m < 0$.

For sequences $\mathbf{x} = (x_k)_{k \in \mathbb{N}}$, $\mathbf{y} = (y_k)_{k \in \mathbb{N}} \subset \mathbb{C}$, let

$$A_{\mathbf{x}, \mathbf{y}} := N + \sum_{k > 0} \left((x_k + y_k N) (s^*)^k + s^k (\overline{x_k} + \overline{y_k} N) \right) \quad (31)$$

as an unbounded operator on $\ell^2(\mathbb{N}_0)$ with domain D , and

$$A_{\mathbf{x}, \mathbf{y}}^{\wedge m} := N^{\wedge m} + \sum_{k=1}^{2m} \left(\left(x_k + y_k N^{\wedge m - \lfloor \frac{k}{2} \rfloor} \right) (s^*)^k + s^k \left(\overline{x_k} + \overline{y_k} N^{\wedge m - \lfloor \frac{k}{2} \rfloor} \right) \right). \quad (32)$$

The proof is split into several steps:

Step 1: If $\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}}$, then for each $C \in \mathbb{R}$ there exist sequences $\mathbf{x} = (x_k)$, $\mathbf{y} = (y_k) \subset \mathbb{C}$ and $C' > 0$, such that $|y_k| < C'$ and $T_\psi(A_{\mathbf{x}, \mathbf{y}}) - A_{\mathbf{x}, \mathbf{y}} = C \cdot \mathbb{1}$.

Proof: From the representation of the transition rates in Figure 2 one reads off that for an operator X on $\ell^2(\mathbb{N}_0)$ with domain D and for $n > 0$, $k \geq 0$ we have

$$\begin{aligned} T_\psi(X)_{n, n+k} &= \alpha^2 X_{n, n+k} + (1 - \lambda) \beta^2 X_{n-1, n+k-1} + \lambda \beta^2 X_{n+1, n+k+1} \\ &\quad + \alpha \beta \nu (X_{n, n+k-1} - X_{n+1, n+k}) + \alpha \beta \bar{\nu} (X_{n-1, n+k} - X_{n, n+k+1}). \end{aligned} \quad (33)$$

Hence,

$$(T_\psi(X) - X)_{n,n+k} = (1 - \lambda)\beta^2(X_{n-1,n+k-1} - X_{n,n+k}) + \lambda\beta^2(X_{n+1,n+k+1} - X_{n,n+k}) \\ + \alpha\beta\nu(X_{n,n+k-1} - X_{n+1,n+k}) + \alpha\beta\bar{\nu}(X_{n-1,n+k} - X_{n,n+k+1}) . \quad (34)$$

Inserting $X = A_{\mathbf{x},\mathbf{y}}$, one finds that $T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}} \stackrel{!}{=} C \cdot \mathbb{1}$ implies for $k > 0$,

$$0 \stackrel{!}{=} (1 - \lambda)\beta^2(-y_k) + \lambda\beta^2 y_k + \alpha\beta\nu(-y_{k-1}) + \alpha\beta\bar{\nu}(-y_{k+1}) \\ \implies y_{k+1} = \frac{2\lambda - 1}{\bar{\nu}} \frac{\beta}{\alpha} y_k - \frac{\nu}{\bar{\nu}} y_{k-1} . \quad (35)$$

The roots of the characteristic polynomial $x^2 - \frac{2\lambda-1}{\bar{\nu}} \frac{\beta}{\alpha} x + \frac{\nu}{\bar{\nu}}$ of the recurrence relation for \mathbf{y} are given by

$$x_{1/2} = \frac{1}{2} \left(\frac{2\lambda - 1}{\bar{\nu}} \frac{\beta}{\alpha} \pm \sqrt{\frac{(2\lambda - 1)^2}{\bar{\nu}^2} \frac{\beta^2}{\alpha^2} - 4 \frac{\nu}{\bar{\nu}}} \right) \\ = \frac{2\lambda - 1}{2\bar{\nu}} \frac{\beta}{\alpha} \left(1 \pm \sqrt{1 - \frac{4|\nu|^2 \alpha^2}{(2\lambda - 1)^2 \beta^2}} \right) . \quad (36)$$

As

$$\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}} \Leftrightarrow \beta^2 \left(1 + \frac{4|\nu|^2}{(2\lambda - 1)^2} \right) < \frac{4|\nu|^2}{(2\lambda - 1)^2} \Leftrightarrow 1 > \frac{4|\nu|^2 \alpha^2}{(2\lambda - 1)^2 \beta^2} , \quad (37)$$

the discriminant is negative, and therefore the absolute values of the roots x_1, x_2 are

$$|x_{1/2}|^2 = \left(\frac{2\lambda - 1}{2\bar{\nu}} \frac{\beta}{\alpha} \right)^2 \left(1 + \frac{4|\nu|^2 \alpha^2}{(2\lambda - 1)^2 \beta^2} - 1 \right) = 1 . \quad (38)$$

Now the general solution to the recurrence (35) for \mathbf{y} , given by $y_k = x_1^k \cdot y' + x_2^k \cdot y''$ for some $y', y'' \in \mathbb{C}$, shows that the sequence \mathbf{y} is necessarily bounded.

Inserting $X = A_{\mathbf{x},\mathbf{y}}$ into (34), but now putting $k = 0$, one finds that $T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}} = C \cdot \mathbb{1}$ implies

$$C + (1 - 2\lambda)\beta^2 = -2\alpha\beta \cdot \Re(\bar{\nu}y_1) . \quad (39)$$

Conversely, choosing y_1 such that this equation holds, and y_k , $k > 1$, via the recurrence (35), we see that $(T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}})_{n,n+k} = (C \cdot \mathbb{1})_{n,n+k}$ holds for all $n > 0$ and $k \geq 0$. The equations $(T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}})_{0,k} \stackrel{!}{=} 0$, $k > 0$, lead to (and are solved by) the following recurrence relation for \mathbf{x} :

$$x_k \stackrel{!}{=} (\lambda\alpha^2 + (1 - \lambda)\alpha)x_k + \alpha\beta\nu(-y_{k-1}) + \lambda\beta^2(x_k + y_k) - \beta\bar{\nu}x_{k+1} \\ \iff x_{k+1} = \frac{1}{\beta\bar{\nu}} ((1 - \lambda)(\alpha - 1)x_k + \lambda\beta^2 y_k - \alpha\beta\nu y_{k-1}) . \quad (40)$$

The condition $(T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}})_{0,0} \stackrel{!}{=} 0$ reads

$$0 \stackrel{!}{=} \lambda\beta^2 - 2\beta\Re(\bar{\nu}x_1) . \quad (41)$$

Hence, choosing the sequences \mathbf{x}, \mathbf{y} according to the initial conditions and recurrence relations just given, the equations $(T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}})_{n,n+k} = (C \cdot \mathbb{1})_{n,n+k}$ are satisfied for all $n, k \geq 0$. The equations $(T_\psi(A_{\mathbf{x},\mathbf{y}}) - A_{\mathbf{x},\mathbf{y}})_{n+k,n} = (C \cdot \mathbb{1})_{n+k,n}$ hold automatically, as they are the complex conjugates of the former.

Step 2: Let $\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}}$, fix $C > 0$, and let \mathbf{x}, \mathbf{y} be as above. If $\frac{1-\alpha}{|\beta|} < \frac{|\nu|}{1-\lambda}$, then there exists $c > 0$ such that $\|T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}\| < 2c$ holds for all $m \in \mathbb{N}$.

Proof: One finds that the matrix entries $(T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]})_{j,k}$ vanish unless $j+k \in \{2m, 2m+1\}$. If $\frac{1-\alpha}{|\beta|} < \frac{|\nu|}{1-\lambda}$, then not only the sequence \mathbf{y} is bounded, but so is the sequence \mathbf{x} , as one sees by inspection of the recurrence relation (40). As the non-vanishing matrix entries of $T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}$ are expressed in terms of simple linear combinations of the x_k, y_k , one easily sees that the matrix entries of $T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}$ are also bounded by some $c > 0$. Since the non-vanishing matrix entries of $T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}$ are concentrated on two (anti-)diagonals, the operator-norm of $T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}$ is bounded by $2c$.

Step 3: If $\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}}$ and $\frac{1-\alpha}{|\beta|} < \frac{|\nu|}{1-\lambda}$, then T_ψ has no normal invariant state.

Proof: Let φ be a normal state on $\mathcal{B}(\ell^2(\mathbb{N}_0))$, fix some $C > 0$, and let $\mathbf{x}, \mathbf{y}, c$ be as above. Choose $\varepsilon > 0$ such that $C \cdot (1 - \varepsilon) > 4\sqrt{\varepsilon}c$, and $m \in \mathbb{N}$ such that $\varphi(p_{[0,m-1]}) > 1 - \varepsilon$. Then, abbreviating $Z := T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - A_{\mathbf{x},\mathbf{y}}^{\wedge m} - C \cdot p_{[0,m-1]}$ and using $Z = Zp_{[0,m-1]}^\perp + p_{[0,m-1]}^\perp Zp_{[0,m-1]}$, we see

$$\begin{aligned} |\varphi(T_\psi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}) - \varphi(A_{\mathbf{x},\mathbf{y}}^{\wedge m}))| &\geq |\varphi(C \cdot p_{[0,m-1]})| - |\varphi(Z)| \\ &> C \cdot (1 - \varepsilon) - \underbrace{|\varphi(Zp_{[0,m-1]}^\perp)|}_{\geq -\sqrt{\varphi(ZZ^*)\varphi(p_{[0,m-1]}^\perp)}} - \underbrace{|\varphi(p_{[0,m-1]}^\perp Zp_{[0,m-1]})|}_{\geq -\sqrt{\varphi(p_{[0,m-1]}^\perp)\varphi(Z^*Z)}} \\ &\geq C \cdot (1 - \varepsilon) - 4\sqrt{\varepsilon}c > 0 . \end{aligned} \quad (42)$$

Hence, φ cannot be invariant. \square

A straightforward calculation shows that $\beta^2 < \frac{1}{1 + \frac{(1-2\lambda)^2}{4|\nu|^2}}$ is equivalent to $\frac{1}{2} - |\frac{\alpha}{\beta}\nu| < \lambda < \frac{1}{2} + |\frac{\alpha}{\beta}\nu|$. It is easy to see that for $|\zeta| = 1$, the condition $\frac{1-\alpha}{|\beta|} < \frac{|\nu|}{1-\lambda}$ is equivalent to $\lambda > \frac{1}{2}(1 - \alpha)$. Hence, as a consequence of the previous proposition and the non-existence statement in Proposition 8, we find that for pure states ψ with $\lambda \neq \frac{1}{2}(1 \pm \alpha)$, the toy model transition operator T_ψ admits only pure invariant states.

To summarize, the regions of the parameter space for the “atomic” state ψ where T_ψ admits an invariant normal state and those where it does not are shown in Figure 3.

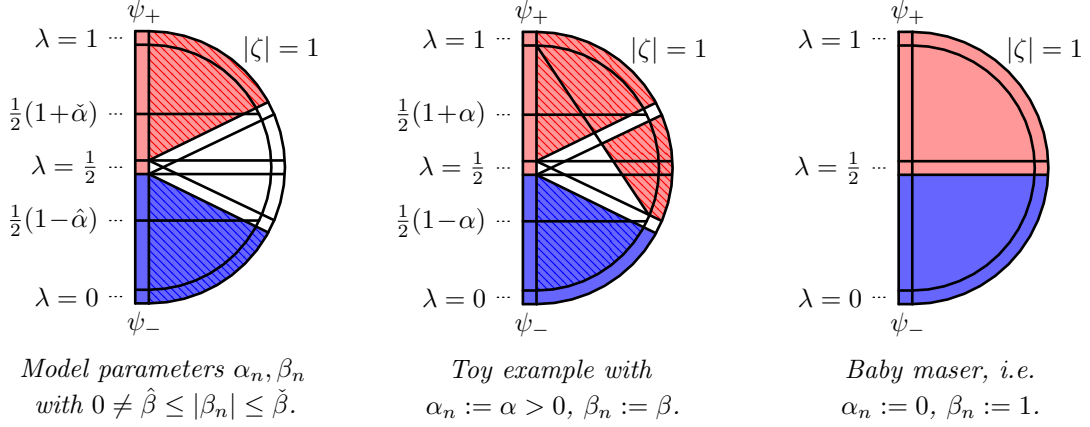


Figure 3: Blue areas indicate that there is an invariant normal state for T_ψ , red areas indicate that there is no such state. Hatched areas are new compared with [BGK⁺13, Fig. 6.2]. In case of the toy example, there are no *pure* invariant states at $\lambda = \frac{1}{2}(1 \pm \alpha)$. Other regions for which we do not have any results are left blank.

5 Example: one-atom maser with random interaction time

The evolution of an electromagnetic mode inside a perfect (no energy loss) cavity which interacts sequentially, according to the Jaynes-Cummings model, during time-intervals of length $\tau > 0$ with two-level atoms prepared in a state ψ is described by the generalised one-atom maser with parameters α_n, β_n chosen as (cf. [BGK⁺13, Ex. 3.4])

$$\alpha_n(\tau) = \cos(g\tau\sqrt{n}) , \quad \beta_n(\tau) = -\sin(g\tau\sqrt{n}) . \quad (43)$$

To make the dependence on τ explicit, we denote the corresponding transition operator by $T_{\psi,\tau}$. As shown in [BP09, Thm. 3.3], this model admits invariant states if ψ is a “thermal state” (i.e. $\nu = 0$) with $\lambda < \frac{1}{2}$. If $\beta_n(\tau) \neq 0$ for all $n > 0$, referred to as the “non-resonant” case in [BP09], then the invariant state is unique, and is given by $\varphi(e_{n,m}) = \delta_{n,m} \cdot \frac{1-2\lambda}{1-\lambda} \left(\frac{\lambda}{1-\lambda}\right)^n$. In this situation, the invariant state is also *absorbing*, meaning that for all normal states θ and observables $x \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ one has $\lim_n \theta(T_{\psi,\tau}^n(x)) = \varphi(x)$; see [Bru13, Thm. 3.2] (the notions “mixing”, defined there, and “absorbing” coincide for faithful states). However, the model exhibits pathological behaviour: For generic $\tau > 0$, the values of $|\beta_n(\tau)|$ come arbitrarily close to 0. Nesting intervals, one can show:

- For a fixed *pure* atomic state ψ with $\lambda < \frac{1}{2}$, the set of values of τ , for which $T_{\psi,\tau}$ admits no *pure* invariant state, is dense in $(0, \infty)$.
- For a fixed *diagonal* atomic state ψ with $\lambda < \frac{1}{2}$, the set of values of τ , for which there is a normal state $\tilde{\varphi}$ with $\text{supp } \tilde{\varphi} \leq p_{[0,m]}$ for some $m \in \mathbb{N}_0$ and $((T_{\psi,\tau})_*^n(\tilde{\varphi}))_{n \in \mathbb{N}_0}$ converging slower than geometrically towards the invariant state, is dense in $(0, \infty)$.

It was suggested in [BJM13, Open problem 3] that adding some randomness might cure the problems. Therefore, we consider here the model where τ is randomly distributed according to some probability measure ρ on $[0, \infty)$. The evolution of the electromagnetic mode is then described by the ucp-map

$$T_{\psi, \rho}(x) := \int_{[0, \infty)} T_{\psi, \tau}(x) d\rho(\tau) , \quad x \in \mathcal{B}(\ell^2(\mathbb{N}_0)) . \quad (44)$$

For a Hilbert space \mathcal{H} , denote by $\text{UCP}(\mathcal{B}(\mathcal{H}))$ the set of ucp-maps on $\mathcal{B}(\mathcal{H})$. The following lemma will be useful:

Lemma 10. *Let μ be a probability measure on a measure space Ω , let \mathcal{H} be a separable Hilbert space, and let $T : \Omega \rightarrow \text{UCP}(\mathcal{B}(\mathcal{H}))$, $\omega \mapsto T_\omega$ be a function such that $\omega \mapsto \langle \xi, T_\omega(x)\xi \rangle$ is measurable for each $\xi \in \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$. A projection $p \in \mathcal{B}(\mathcal{H})$ is subharmonic for $\int_\Omega T_\omega d\mu(\omega)$ if and only if p is subharmonic for T_ω for almost all $\omega \in \Omega$.*

Proof. Recall that a projection p is subharmonic for a ucp-map T if and only if $T(p) \geq p$, if and only if $T(p^\perp) \leq p^\perp$, if and only if $pT(p^\perp)p = 0$. For each $\xi \in \mathcal{H}$ we have $\langle p\xi, \int_\Omega T_\omega(p^\perp) d\mu(\omega)p\xi \rangle \stackrel{(\text{by def})}{=} \int_\Omega \langle \xi, pT_\omega(p^\perp)p\xi \rangle d\mu(\omega) = 0$ if and only if $pT_\omega(p^\perp)p\xi = 0$ for almost all ω , since $pT_\omega(p^\perp)p \geq 0$. As \mathcal{H} is separable, we have $p \int_\Omega T_\omega(p^\perp) d\mu(\omega)p = 0$ if and only if $pT_\omega(p^\perp)p = 0$ for almost all ω . \square

Proposition 11. *Let $\rho = D_\rho(\tau)d\tau$ be a probability measure with density $D_\rho \in C^1([0, \infty))$ such that $\frac{d}{d\tau}D_\rho \in L^1([0, \infty))$. Let $0 \leq \lambda \leq 1$. Then:*

1. *$T_{\psi, \rho}$ has a normal invariant state if $\lambda < \frac{1}{2}$; if $\lambda > \frac{1}{2}$, then $T_{\psi, \rho}$ has no normal invariant state.*
2. *If $\lambda < \frac{1}{2}$ and ψ is faithful, then the normal invariant state for $T_{\psi, \rho}$ is absorbing.*

Proof. First, the parameters μ_n, λ_n, η_n of $T_{\psi, \rho}$ are given by

$$\begin{aligned} \mu_n &= \int_{(0, \infty)} \text{Tr}(e_{n, n} T_{\psi, \tau}(e_{n-1, n-1})) d\rho(\tau) = (1 - \lambda) \int_{(0, \infty)} \sin^2(g\tau\sqrt{n}) d\rho(\tau) , \\ \lambda_n &= \int_{(0, \infty)} \text{Tr}(e_{n, n} T_{\psi, \tau}(e_{n+1, n+1})) d\rho(\tau) = \lambda \int_{(0, \infty)} \sin^2(g\tau\sqrt{n+1}) d\rho(\tau) , \\ \eta_n &= \int_{(0, \infty)} \text{Tr}(e_{n, n+1} T_{\psi, \tau}(e_{n, n})) d\rho(\tau) = \bar{\nu} \int_{(0, \infty)} \sin(g\tau\sqrt{n+1}) \cos(g\tau\sqrt{n+1}) d\rho(\tau) \\ &= \frac{\bar{\nu}}{2} \int_{(0, \infty)} \sin(2g\tau\sqrt{n+1}) d\rho(\tau) , \end{aligned} \quad (45)$$

cf. Figure 2. As $n \rightarrow \infty$, they approach the limits $\lim_n \mu_n = \frac{1}{2}(1 - \lambda)$, $\lim_n \lambda_n = \frac{\lambda}{2}$ and

$\lim_n \eta_n = 0$: for example,

$$\left| \frac{2}{\nu} \eta_n \right| = \left| \int_{(0,\infty)} \sin(2g\tau\sqrt{n+1}) d\rho(\tau) \right| \leq \frac{1}{2g\sqrt{n+1}} \left\{ \underbrace{\left| \cos(2g\tau\sqrt{n+1}) D_\rho(\tau) \right|_0^\infty}_{\leq D_\rho(0)} + \underbrace{\int_{(0,\infty)} \left| \cos(2g\tau\sqrt{n+1}) \left(\frac{d}{d\tau} D_\rho \right)(\tau) \right| d\tau}_{\leq \left\| \frac{d}{d\tau} D_\rho \right\|_1} \right\} \xrightarrow{n \rightarrow \infty} 0. \quad (46)$$

An application of Theorem 7 along the lines of the proof of Proposition 8 shows part 1.

For part 2, observe that $T_{\psi,\rho}$ is irreducible by Lemma 10, since almost all $T_{\psi,\tau}$ are irreducible. Hence, the invariant state for $T_{\psi,\rho}$, existing thanks to part 1, is faithful. Moreover, if ψ_- is the state on M_2 corresponding to $\lambda = 0$, then $T_{\psi,\rho}$ is a non-trivial convex combination of $T_{\psi_-,\rho}$ with some other ucp-map. As the invariant state $x \mapsto \langle x e_0, e_0 \rangle$ of $T_{\psi_-,\rho}$ is absorbing, $T_{\psi,\rho}$ has an absorbing state by Theorem 12 or by Proposition 13. \square

Appendix: Approach to equilibrium

A normal invariant state φ is called *absorbing* for a ucp-map T , if for each normal state θ and for all observables x we have $\lim_n \theta(T^n(x)) = \varphi(x)$. The following result is used in the proof of Proposition 11 above:

Theorem 12 ([Haa06, Kor. 2.2.11 & Satz 2.4.9]). *Let R, S be ucp-maps on $\mathcal{B}(\mathcal{H})$, and let $T = \lambda R + (1 - \lambda)S$ for some $0 < \lambda \leq 1$. Suppose that T admits a faithful invariant normal state. Then if R has an absorbing state, so does T .*

As the proof given in [Haa06] is available in German language only, and as the author is unaware of another reference for this statement, we will here give a quantitative variant of Theorem 12. This allows to give a slightly shorter proof, is sufficient for the purposes above, and might be of independent interest.

Let g be a function $\mathbb{N} \rightarrow \mathbb{R}$. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is said to be of order g , written $f \in \mathcal{O}(g)$, if there is $C > 0$ such that $|f(n)| < Cg(n)$ for all $n \in \mathbb{N}$.

Proposition 13. *Let R, S be ucp-maps on $\mathcal{B}(\ell^2(\mathbb{N}_0))$, and let $T = \lambda R + (1 - \lambda)S$ for some $0 < \lambda < 1$. Moreover, suppose that T admits a faithful normal invariant state φ of exponential fall-off: $\sum_{k \geq n} \varphi(e_{k,k}) \in \mathcal{O}(e^{-\gamma_2 n})$. Then if R has an absorbing state φ_R such that $\|\theta \circ R^n - \varphi_R \circ \text{Ad}_{p_{[0,m]}}\| \leq e^{\gamma_0 m - \gamma_1 n}$ holds for all normal states θ with $\text{supp } \theta \leq p_{[0,m]}$, $m \in \mathbb{N}$, then T has an absorbing state φ_T such that $\|\theta \circ T^n - \varphi_T\| \in \mathcal{O}(n^{-\gamma(a)})$, where $\gamma(a) = \frac{\gamma_1 \gamma_2}{-a \ln \lambda \cdot (\gamma_0 + \gamma_2)}$, holds for all $a > 1$ and for all θ with $\text{supp } \theta \leq p_{[0,m]}$ for some $m \in \mathbb{N}$.*

Proof. (along the lines of the proof of [Haa06, Satz 2.4.9]) The idea is to show that $\sup_{N_1, N_2 \geq N} \|\theta \circ T^{N_1} - \theta \circ T^{N_2}\| \in \mathcal{O}(N^{-\gamma(a)})$.

For words \vec{i} over $\{0, 1\}$, recursively define ucp-maps $T^{\vec{i}}$ by putting $T^\emptyset := \text{id}$, $T^{\vec{i}0} := T^{\vec{i}} \circ R$ and $T^{\vec{i}1} := T^{\vec{i}} \circ S$. Let μ be the probability measure on $\{0, 1\}^N$ with $\mu(\{\vec{i}\}) := \lambda^n (1 -$

$\lambda)^{N-n}$, if \vec{i} is a word of length N containing the letter “0” n -times. Then we have $\theta \circ T^N = \sum_{\vec{i} \in \{0,1\}^N} \mu(\{\vec{i}\}) \cdot \theta \circ T^{\vec{i}}$. Denote by $\mathfrak{R}_r^N \subseteq \{0,1\}^N$ the set of words containing at least one run of r consecutive 0’s. We have $1 - \mu(\mathfrak{R}_r^N) \leq (1 - \lambda^r)^{\lfloor \frac{N}{r} \rfloor}$ (divide $\{1, 2, \dots, N\}$ into $\lfloor \frac{N}{r} \rfloor$ blocks of length r and only count runs fitting into one of these blocks). Hence, for $N, N_1, N_2 \in \mathbb{N}$ with $N_1, N_2 \geq N$,

$$\begin{aligned} \|\theta \circ T^{N_1} - \theta \circ T^{N_2}\| &\leq \left\| \sum_{\vec{i} \in \mathfrak{R}_r^N} \mu(\{\vec{i}\}) \left(\theta \circ T^{N_1-N} \circ T^{\vec{i}} - \theta \circ T^{N_2-N} \circ T^{\vec{i}} \right) \right\| \\ &\quad + \underbrace{\left\| \sum_{\vec{i} \notin \mathfrak{R}_r^N} \mu(\{\vec{i}\}) \left(\theta \circ T^{N_1-N} \circ T^{\vec{i}} - \theta \circ T^{N_2-N} \circ T^{\vec{i}} \right) \right\|}_{\leq 2(1-\lambda^r)^{\lfloor \frac{N}{r} \rfloor}}. \end{aligned} \quad (47)$$

Let θ be a normal state with $\text{supp } \theta \leq p_{[0,M]}$. Since φ is faithful, there exists $C_1 > 0$ with $\theta \leq C_1 \varphi$. Since φ falls off exponentially, there are $C_2, \gamma_2 > 0$ such that $\varphi(p_{[0,M]}^\perp) < C_2^2 e^{-2\gamma_2 M}$ for all $M \in \mathbb{N}$. Then we have, for $x \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ and $M \in \mathbb{N}$,

$$\begin{aligned} &\left| \sum_{\vec{i} \in \mathfrak{R}_r^N} \mu(\{\vec{i}\}) \left(\theta \circ T^{N_1-N} \circ T^{\vec{i}} - \theta \circ T^{N_2-N} \circ T^{\vec{i}} \right) (x) \right| \\ &= \left| \sum_{k=r}^N \sum_{\vec{i} \in \{0,1\}^{k-r}} \mu(\{\vec{i}\}) \left(\theta \circ T^{N_1-k} \circ R^r \circ T^{\vec{i}} - \theta \circ T^{N_2-k} \circ R^r \circ T^{\vec{i}} \right) (x) \right| \\ &\leq \sum_{k=r}^N \sum_{\vec{i} \in \{0,1\}^{k-r}} \mu(\{\vec{i}\}) \left(\left| \left(\theta \circ T^{N_1-k} - \theta \circ T^{N_2-k} \right) (p_{[0,M]} R^r (T^{\vec{i}}(x)) p_{[0,M]}) \right| \right. \\ &\quad + \left| \left(\theta \circ T^{N_1-k} - \theta \circ T^{N_2-k} \right) (p_{[0,M]}^\perp R^r (T^{\vec{i}}(x)) p_{[0,M]}) \right| \\ &\quad + \left| \left(\theta \circ T^{N_1-k} - \theta \circ T^{N_2-k} \right) (p_{[0,M]} R^r (T^{\vec{i}}(x)) p_{[0,M]}^\perp) \right| \\ &\quad \left. + \left| \left(\theta \circ T^{N_1-k} - \theta \circ T^{N_2-k} \right) (p_{[0,M]}^\perp R^r (T^{\vec{i}}(x)) p_{[0,M]}^\perp) \right| \right) \\ &\stackrel{(*)}{\leq} 2C_1 C_2 e^{-\gamma_2 M} \|x\| \\ &\leq \sum_{k=r}^N \sum_{\vec{i} \in \mathfrak{R}_r^k} \mu(\{\vec{i}\}) \left(\underbrace{\left| \left(\theta \circ T^{N_1-k} - \varphi_R \right) (p_{[0,M]} R^r (T^{\vec{i}}(x)) p_{[0,M]}) \right|}_{\leq e^{\gamma_0 M - \gamma_1 r} \|x\|} \right. \\ &\quad \left. + \underbrace{\left| \left(\theta \circ T^{N_2-k} - \varphi_R \right) (p_{[0,M]} R^r (T^{\vec{i}}(x)) p_{[0,M]}) \right|}_{\leq e^{\gamma_0 M - \gamma_1 r} \|x\|} \right) + 6C_1 C_2 e^{-\gamma_2 M} \|x\| \\ &\leq (2C^M g(r) + 6C_1 C_2 e^{-\gamma_2 M}) \|x\|. \end{aligned} \quad (48)$$

where for (*) the Cauchy-Schwarz inequality was used. Altogether, we have

$$\|\theta \circ T^{N_1} - \theta \circ T^{N_2}\| \leq 2Ce^{\gamma_0 M - \gamma_1 r} + 6C_1 C_2 e^{-\gamma_2 M} + 2(1 - \lambda^r)^{\lfloor \frac{N}{r} \rfloor}. \quad (49)$$

Now, choosing $r := \lfloor \frac{\ln N}{a \ln \lambda} \rfloor$ for some $a > 1$, one finds $(1 - \lambda^r)^{\lfloor \frac{N}{r} \rfloor} = \exp(\lfloor \frac{N}{r} \rfloor \cdot \ln(1 - \lambda^r)) \leq \exp(-\lfloor \frac{N}{r} \rfloor \cdot \lambda^r) \in \mathcal{O}(\exp(\frac{N}{a \ln \lambda} e^{\ln \lambda \frac{\ln N}{-a \ln \lambda}})) = \mathcal{O}(\exp(\frac{N^{1-\frac{1}{a}}}{a \ln \lambda}))$. Moreover, putting $M := \lfloor \frac{\gamma_1}{\gamma_0 + \gamma_2} \frac{\ln N}{-a \ln \lambda} \rfloor$, we obtain $e^{\gamma_0 M - \gamma_1 r}, e^{-\gamma_2 M} \in \mathcal{O}(\exp(-\frac{\gamma_1 \gamma_2}{\gamma_0 + \gamma_2} \frac{\ln N}{-a \ln \lambda})) = \mathcal{O}(N^{-\gamma(a)})$, where $\gamma(a) = \frac{\gamma_1 \gamma_2}{-a \ln \lambda \cdot (\gamma_0 + \gamma_2)}$. \square

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